THE TREATMENT OF LINEAR FUNCTIONS IN JAPANESE CURRICULA THROUGH QUANTITATIVE AND COVARIATIONAL REASONING

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This study investigated how Japanese curricula represent functional relationships through the lenses of quantitative and covariational reasoning. Utilizing both macro and micro textbook analyses, we examined the tasks, questions, and representations in the Japanese elementary and lower secondary course of study, teachers' guide, and textbooks. Findings showed that, starting from the 4th grade by the end of 8th grade, Japanese curricula focus on iteratively improving learners' quantitative and covariational reasoning gradually raising up to continuous covariation level. This pointed out that Japanese curricula have a spiral nature in the learning of functional relationships involved in proportional and linear situations. We discuss the implications of findings for teaching, learning, and teacher education.

Keywords: curriculum, quantitative reasoning, covariational reasoning, linear functions

Background

School curriculum has a role in improving students' learning and acvement (Cai, 2017). The term curriculum is a broad notion including intended, implemented, attained, and potentially implemented curriculum. *Intended curriculum* is a planned set of actions for students to achieve a specific goal. *Implemented curriculum* refers to how the goals of the intended curriculum are implemented within classrooms and the *attained curriculum* is considered as what is learned by students in classrooms. Within the span of curriculum research, textbooks are considered a potentially *implemented curriculum* as they build a pathway "… between the intentions of the designers of curriculum policy and the teachers that provide instruction in classrooms" (Valverde et al., 2002, p.2). Being mediators between intended and implemented curriculum, textbooks are considered as having a strong impact on what occurs in classrooms (Valverde et al., 2002). In addition, textbooks provide information about pedagogical predispositions, nature of subject matter, arrangements of topics, and level of complexity. Therefore, textbooks have been accepted as tools to be investigated (Cai, 2017) in order to gain some possible insights about opportunities for students to learn mathematics (e.g., Son & Senk, 2010), teacher's teaching and learning (e.g., Son & Kim, 2015), and educational preferences of a particular country (e.g., Usiskin, 2013). Son and Diletti (2017) further argued that teachers' guidebooks and other supplementary materials (e.g., assessments) might be classified as potentially implemented curriculum since they also have potential impact on teaching. It is in this regard that, in this study, we examined Japanese curricula including the course of study, teachers' guide and textbooks because of their potential impact on learning of mathematics topics and classroom teaching.

Textbook analysis is mainly done either on the overall structure of textbooks, which is named as macroanalysis (i.e., horizontal analysis) or on the treatment of specific mathematics topics, which is called as microanalysis (i.e., vertical analysis). Researchers recommended to use both the macro and micro analysis to investigate the treatment of a mathematics topic so that the topic could be examined in terms of its place in specific educational system, related topics to it, and learning opportunities provided in it (Charalambous et al., 2010; Watanabe et al., 2017; Li, 2000). Regarding microanalysis, the treatment of functions is one important notion to study

because functions are the backbone of the domain of algebra and lays the foundation for advanced mathematics (Kaput, 1994). Functional relationships between quantities can be modeled by linear, quadratic, polynomial, exponential functions etc. Although scarce in number (Son & Diletti, 2017), there are some studies focusing on secondary school mathematics such as quadratic functions (e.g., Sağlam & Alacacı, 2012) and linear functions (e.g., Wang et al., 2017). Particularly, Wang et al. (2017) examined mathematics textbooks in Shanghai and in England by looking at the hierarchical levels of students' understanding of linear functions through some aspects, such as dependent relationship and connecting representations. Functional relationships can mainly be expressed in two ways: (i) by a correspondence relationship between elements of two sets and (ii) by representing covariation in quantities (Blanton et al., 2011). Though, the topics of functions were suggested to be developed through quantitative and covariational reasoning and to do it starting with earlier grades rather than waiting until high school and college (e.g., Oehrtman et al., 2008). Thompson and Carlson (2017) defined:

A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (p. 436)

Lloyd et al. (2010) pointed to the essentiality of understanding functions as means to outline how related quantities covary. In addition, studies showed that students who cannot reason variationally and covariationally seemed experiencing difficulties about functional and proportional relationships (e.g., Carlson et al., 2002). It is also found that undergraduate students start their collegiate study with weak understanding of functions, thus experiencing deficiency about making sense of dynamic events (e.g., Carlson et al., 2002). Therefore, research suggested designing instruction and curriculum for improving the learners' understanding of functional relationships (Thompson & Carlson, 2017; Carlson et al., 2002). Thus, in compliance with earlier research (e.g., Karagöz et al. 2022; Taşova et al., 2018), we acknowledge that quantitative and covariational reasoning might be a possible avenue for analyzing curricular materials. Similarly, Thompson and Carlson (2017) argue that Japanese curricula exemplifies how quantitative and covariational reasoning can be integrated in standards and textbooks.

It is in this regard that, in this study, we investigated the topic of functional relationships in Japanese curricula between the grades of 4 to 8 through the lenses of quantitative and covariational reasoning. We also particularly attended to the call for integrating macro and micro analyses to investigate the research questions of "How might Japanese curricula materials depicted in the topic of functional relationships potentially trigger quantitative reasoning and covariational reasoning? In what ways tasks, problem situations, questions, and the use of representations (including algebraic, graphical, tabular, and verbal) in functional relationships might potentially trigger quantitative and covariational reasoning in Grades 4 to 8?"

Theoretical Framework

Thompson (1990) defined quantitative reasoning as "the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (p. 12). A quantity is a measurable quality of an object coming into being with a person's conception of a situation by considering the measurable quality of an object (Thompson, 1994a). Both quantitative and covariational reasoning is about conceiving a situation where the former requires a person to conceive the situation in terms of a quantitative structure and the latter is necessary for a person to conceive the dynamic situation which involves quantities' varying values (Thompson &

Carlson, 2017). Carlson et al. (2002) defined covariational reasoning as “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354). They argued that these activities (i.e., images of covariation) are developmental, meaning that “the images of covariation can be defined by level and that the levels emerge in an ordered succession” (Carlson et al., 2002, p. 354). In their covariational reasoning framework, Carlson et al. presented five mental actions originating in five covariational reasoning levels (see Figure 1).

Covariational Reasoning Level	**Description of the Covariational Reasoning Level**	**Necessary Mental Actions for the reasoning level**	**Description of the highest mental action**
Smooth continuous covariation	“The person envisions increases and decreases (thereafter, changes) in one quantity’s or variable’s value, and the person envisions both variables varying smoothly and continuously.” (Thompson and Carlson, 2017, p. 435)	MA5 MA4 MA3 MA2 MA1	**MA5:** “Coordinating the instantaneous rate of change of the functions with continuous changes in the independent variable for the entire domain of the function.” (Carlson *et al*., 2002, p. 357)
Chunky continuous covariation	“The person envisions changes in one variable’s value as happening simultaneously with changes in another variable’s value, and they envision both variables varying with chunky continuous variation.” (Thompson and Carlson, 2017, p. 435)	MA4 MA3 MA2 MA1	**MA4:** “Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.” (Carlson *et al*., 2002, p. 357)
Coordination of values	“The person coordinates the value of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x,y).” (Thompson and Carlson, 2017, p. 435)	MA3 MA2 MA1	**MA3:** “Coordinating the amount of change of one variable with changes in the other variable.” (Carlson *et al*., 2002, p. 357)
Gross coordination of values	“The person forms a gross image of quantities’ values varying together, such as “this quantity increases while that quantity decreases.” The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities’ values.” (Thompson and Carlson, 2017, p. 435)	MA2 MA1	**MA2:** “Coordinating the direction of change of one variable with changes in the other variable.” (Carlson *et al*., 2002, p. 357)
Precoordination of values	“The person envisions two variables’ values varying, but asynchronously- one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.” (Thompson and Carlson, 2017, p. 435)	MA1	**MA1:** “Coordinating the value of one variable with changes in the other.” (Carlson *et al*., 2002, p. 357)
No coordination	“The person has no image of variables varying together. The person focuses on one or another variable’s variation with no coordination of values.” (Thompson and Carlson, 2017, p. 435)	-	-

Figure 1: Covariational reasoning levels and corresponding mental actions

Method

In this study we examined the Mathematics International (MI) textbook series published in 2011 globally by Tokyo Shoseki in collaboration with Global Educational Resources. Tokyo Shoseki is a leading textbook publisher in Japan and its textbook series are one of the six most widely used series in elementary mathematics (Watanabe et al., 2017). The MI series cover grades 1 to 9 which includes both elementary (grades 1 to 6) (Fujii & Iitaka, 2012) and lower secondary (grades 7 to 9) school mathematics (Fujii & Matano, 2012). In addition, we investigated the course of study (COS) published in 2008, which is the national curriculum standards in Japan, and teachers’ guides, which is defined as “a guidebook on Japanese curriculum standards” (Isoda, 2010a, p. i) for teachers. Both were provided by the Ministry of National Education of Japan.

To analyze the curricula materials, we utilized a content analysis method (Weber, 1990). First, we determined the grade levels in the Japanese COS where the functional relationships were covered. We found out that under the topic of functional relationships, rate, ratio, proportion, and linear functions were covered between the grades 4 and 8. In this paper we specifically focus on linear functions and proportional relationships. We conducted our analysis in the order of the course of study (COS), teachers’ guide, and the units of textbooks. Our unit(s) of analysis in all the curricular materials were a statement or a set of statements, graphs, diagrams, tables and symbols. Using both quantitative reasoning and covariational reasoning frameworks as our theoretical constructs, we examined how quantities were introduced and how the relationships between them were triggered in the statements, tasks, questions, problem

situations and representations throughout curricula. As covariational reasoning enables examining quantitative reasoning in dynamic situations, using the categories in Figure 1, we examined which mental actions were targeted on the part of students to engage in covariational reasoning. This way, we determined the tasks, questions, problem situations, and representations that seemed potentially to trigger the mental actions, and eventually covariational reasoning.

Findings

Findings has shown that the level of covariational reasoning and complexity of task variables increases with the increase in grade level (see Figure 2).

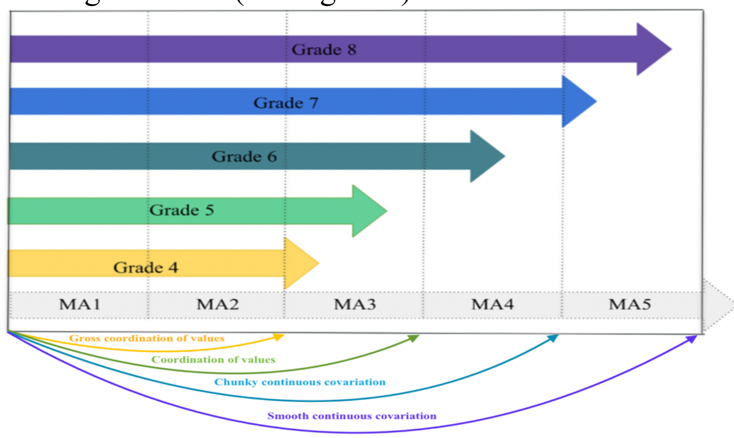

Figure 2: Spiral and iterative nature of Japanese curricula with respect to the metal actions in covariational reasoning

Particularly, in the context of functional relationships, the gross coordination level of covariational reasoning has been supported starting from the 4th grade to the 8th grade. Although the tasks and questions in each grade mostly focus on and delve into specific mental actions with the corresponding covariational reasoning level, there is a spiral nature such that the covariational reasoning level in the upcoming grade builds on and deepens on the previous ones. This suggests that mental actions in the previous covariational reasoning level are iteratively triggered to build the next level of reasoning. Particularly, *gross coordination of values* level of covariational reasoning seems to be aimed in the 4th grade by particularly triggering MA1 and MA2; *coordination of values* level seems to be targeted in the 5th grade by particularly triggering MA1, MA2, and MA3; and *continuous covariation* seems to be targeted in the 6th and 7th grades by particularly triggering MA1, MA2, MA3, and MA4; and (chunky and smooth) *continuous covariation* seems to be targeted in the 8th grade by particularly triggering MA1, MA2, MA3, MA4 and MA5. In lieu of space, we do not share detailed analysis, rather we aim to present different mental actions, reasoning levels, or representations that's not included in the previous grade.

In *4th grade*, in the COS, students are expected to "represent and investigate the relationship between two quantities as they vary simultaneously" (Takahashi et al., 2008, p. 11). Teachers are suggested to use "activities to find two quantities in everyday life that vary in proportion to each other, and to represent and investigate the relationships of numbers/ quantities in tables and graphs." (Isoda, 2010a, p. 116). Also, the Japanese curriculum explicitly differentiates quantities from numbers as stating "a quantity expresses size of an object…A number of objects can be expressed in integers, for example by counting them. On the other hand, in measuring length of

strings or weight of water, the quantity can be divided infinitely and cannot always be expressed integers" (Isoda, 2010, p.34). Particularly, teachers are suggested to present activities having quantities of actual objects compared to make it easier for students "to grasp what quantity is being compared, and the meaning of the quantity will become gradually clear to students …For length, for example, "long/short"; … for speed, "fast/slow" (Isoda, 2010, p. 35). All these suggested that there is emphasis on the fact that quantities are measurable qualities of objects such that speed measures whether an object is fast or slow while length measures whether an object is long or short. In the textbook, students are given a broken-line graph to represent monthly temperature changes of Tokyo to build the gross coordination of values level of covariational reasoning. Particularly, students are expected to verbally indicate the axes and coordinate the values, such as, the highest temperature and the corresponding specific month (MA1). They are also expected to focus on the direction of change (i.e., increase, decrease, constancy) of temperature in months (MA2). Thus, the goal in this grade seems to trigger *gross coordination of values* level of covariational reasoning.

In *5th grade*, the objectives and teachers' guide explanations aim for students to (i) deepen their understanding about covarying quantities, and (ii) consider and express quantitative relationships through using different representations. The main goal seems to deepen the *gross coordination of values* level; but also *coordination of amount of changes* of variables (MA3) has seemed to be introduced. In the textbook, through a task asking to use sticks of the same length side by side to make 30 squares, students are expected to recognize the pattern of an arithmetic sequence by examining the change in quantities and their relationships. As an example of two different students' ideas, the following figure is given in the textbook.

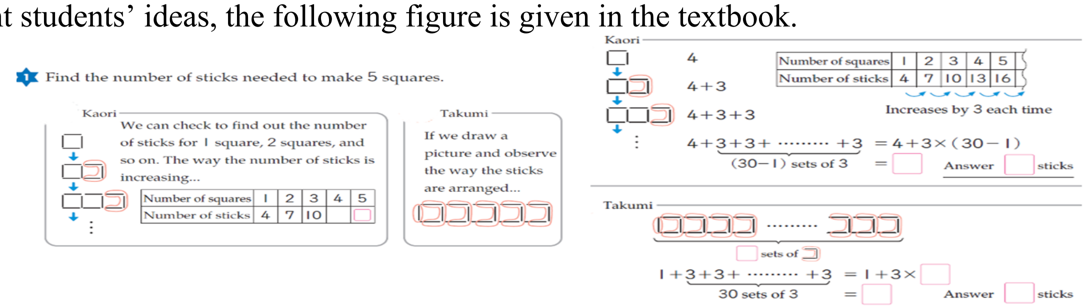

Figure 3: Students' ideas on stick and square task (Fujii and Iitaka, 2012, Grade 5, p. A103-104)

The variables are the number of squares and the number of sticks which are both discrete. A tabular representation is presented in Kaori' approach where she considers the changes between two consecutive values of the number of sticks for each square. However, Takumi examines the arrangement of sticks in the picture where his focus is on how many more sticks (i.e., 3 more sticks) is needed for each square given the first stick. Particularly, Kaori presented quantities, number of squares and number of sticks, in chunks at her table. Albeit the numerical values, her focus is on how the two quantities vary simultaneously such that the number of squares is increasing by one while the number of sticks is increasing by three each time. That is, students' attention is on the increments of 1 and 3 that quantify the increase in the number of squares and the number of sticks respectively. Also, students are expected to compare the differences (the increments of one and three) between the two quantities which might help determine the idea that for each difference (increment of one) in the number of squares, there is some other difference (increment of three) in the number of sticks. Moreover, students are expected to

verbalize what the values of 3, 4, 30, and (30-1) represent given the mathematical sentence $4 + 3 \times (30 - 1)$, an important connection between algebraic and verbal representations. In addition, Kaori's table seems to illustrate the correspondence of increase in number of squares with the increase in number of sticks, and her explanation supports that she considers "the way the number of sticks increasing…" (Fujii and Iitaka, 2012, Grade 5, p. A103). Therefore, Kaori's behavior seems to underscore MA3 and indicate her covariational reasoning level as *coordination of values*. On the contrary, Takumi does not seem to recognize that for each new square there are three more sticks needed. His way of thinking suggests that he seems to focus on the number of squares and the "increase" in the number of sticks rather than coordinating the amount of change in one quantity with the change in the other as Kaori. Takumi's behavior seems to underscore MA2 which corresponds to *gross coordination of values*.

In *6th grade*, students are expected to both (i) understand and investigate direct proportional relationships and (ii) gain an awareness of inversely proportional relationships by examining the situations that involve quantities varying simultaneously (Takahashi et al., 2008). The teachers' guide states that "the quotient of two corresponding quantities remains constant" (Isoda, 2010a, p. 155) in the direct proportional relationships which represent "a straight line passing through the origin." (Isoda, 2010a, p. 155). This characteristic is explicitly stated as important to distinguish proportional relationships. Teachers are suggested to use "activities such as representing on a graph two quantities that vary simultaneously so that they [students] understand that if two quantities are in a proportional relationship, the graph representing this relationship is a straight line" (Isoda, 2010a, p. 155)

The task below from 6th grade textbook shows that *coordination of values* level of covariation (up to MA3) aimed to be deepened and *continuous covariation* (MA4) seemed to be triggered. In lieu of space, we do not provide detailed analysis but our account of how students might possibly reason.

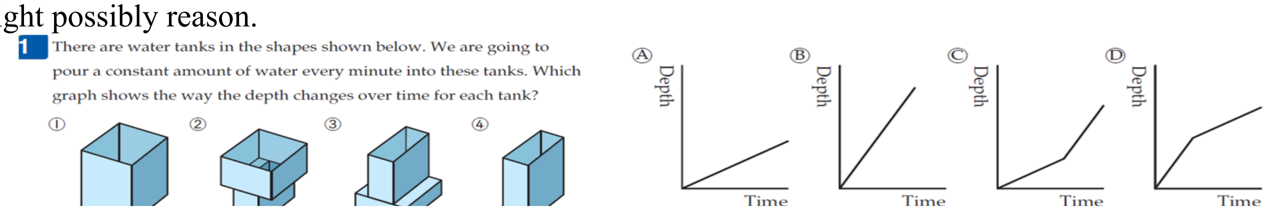

Figure 4: What Will the Graph Look Like (Fujii and Iitaka, 2012, Grade 6, p. B100)

In the task, two continuous variables, time and depth of water, are simultaneously compared through using graphical representations. Students might think that the narrower the base area of the tank the greater the amount of water poured into the tank in a minute. If a student thinks the coordination of depth with amount of time passing (M1), considers the increase in the depth of the water with regard to the time (MA2), compares the amounts of changes in depth of the water for some amount of time (MA3), and envisions the average rate of change of depth as increasing simultaneously with the time (MA4), the student might have *continuous covariation* level of covariational reasoning. However, the action of coordinating the rate of change with the uniform increments of inputs (MA4) are not examined through the numerical values of quantities. Thus, we claim that the task might be targeted to raise students' awareness about the rate of change as a scaffold for the upcoming grades rather than explicitly fostering MA4 in this task.

In *7th grade*, students are expected to further enhance their understanding of direct and inverse proportional relationships in real-life situations. Yet, differently from the 6th grade, the direct and inverse proportional relationships is planned to be re-examined through focusing on

the simple forms of linear functions $y = ax$ and reciprocal functions $y = \frac{a}{x}$ respectively (Isoda, 2010b). Differently from 6th grade, students are expected to examine similarities and differences between direct and inverse proportion graphs. Moreover, the rationale of the use of proportional relationships is explicitly stated as to trigger correspondence and covariation meaning of functions in many phenomena in daily life (Isoda, 2010b).

We share an example from the MI textbook showing direct and indirect proportional relationship between two variables and how the continuous nature of their covarying relationship is represented visually (see Figure 5).

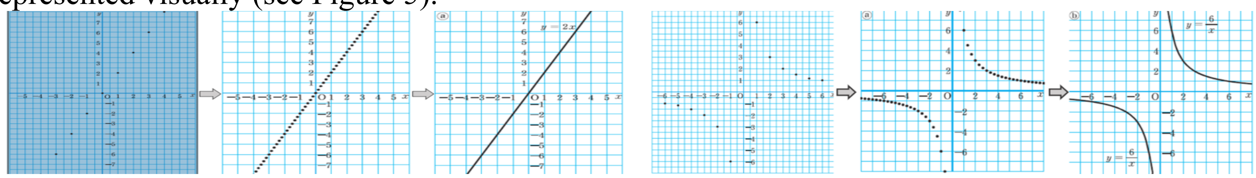

Figure 5: Graphing of y=2x (Fujii and Matano, 2012, Grade 7, p. 117-118) and graphing of $y = \frac{6}{x}$ (Fujii and Matano, 2012, Grade 7, pp. 127-129)

Different integer values of y and x are given for plotting both $y = 2x$ and $y = \frac{6}{x}$. Students are asked to explicitly think about the interval for x values and corresponding y values getting smaller and smaller producing a straight line or a curve. It seems that the continuous nature of variables is targeted through graphical representations. This way of representing might lay a foundation to smooth continuous covariation (corresponding up to MA5), although it is not explicitly introduced.

In *8th grade*, linear functions are introduced, and continuous covariation seems to be triggered and deepened. Students are expected to examine the changes and correspondences of two quantities again with a focus on graphical, algebraic, tabular, and verbal representations. In addition, the rate of change for the linear functions (i.e., in the form of $y = ax + b$) is expressed as $\frac{y_2 - y_1}{x_2 - x_1}$. Here, the rate of change which equals the constant a, is expressed as "how much y will increase when x increases by 1" (Isoda, 2010b, p. 92). All these focus on relationships between amounts of changes in variables suggested that at least MA3 is targeted. Moreover, there is an emphasis on the difference between the expressions of equations with two variables and functions. When the coordination of values of y and x is considered for an equation with two variables expressed as $ax + by + c = 0$, there is one and only one y value for every x value if $b \neq 0$. This relationship indicates that y is a function of x and the equation could be rewritten as $y = -\frac{a}{b}x - \frac{c}{b}$ to illustrate the functional relationship explicitly (Isoda, 2010b).

We present an example from the textbook (see Figure 6) in which the values of temperature are decimal numbers not increasing uniformly with uniform integer changes in time.

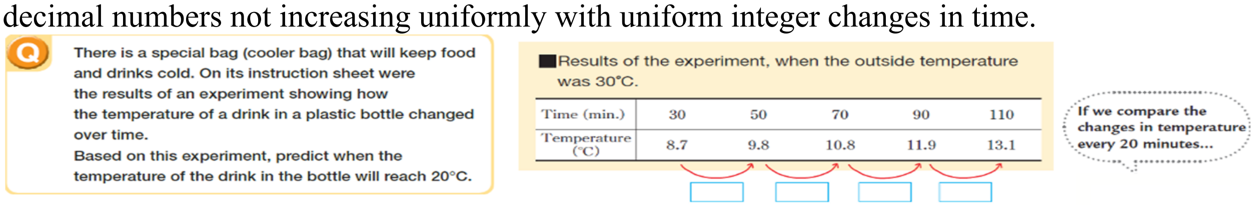

Figure 6: Temperature change of drinks task (Fujii and Matano, 2012, Grade 8, p. 70)

Students are asked to represent this situation with a graph on a coordinate system and think of the relationships between variables where temperature is a function of time. Students are

expected to use tables to express the result of values of variables and placing the values of temperature for corresponding units of time on a coordinate system (MA1), verbalize the direction of change as increase or decrease (MA2), coordinate the amount of change in y with the amount of increase in x (MA3), and coordinate the average rate of change of the temperature with uniform changes in time (MA4). Thus, students are supported to reason on the *chunky continuous covariation* level. Yet, in case students consider the rate of change as an instantaneous rate of change, MA5 might also be triggered. In 8th grade, students are also asked to compare linear functions (i.e., $y = ax + b$) with proportional equations (i.e., $y = ax$). It is explicitly explained that translation of the graph of $y = ax$ on the y axis by the value of b represents $y = ax + b$, where the value of b is expressed as y-intercept. Moreover, the use of different types of representations and the relationship between them are explicitly presented.

Discussion and Conclusion

The findings showed that proportional and functional relationships are presented as intertwined with the quantitative and covariational reasoning starting from 4th grade to 8th grade in Japanese curricula. Starting from gross coordination of values, the highest level of covariational reasoning (smooth covariational reasoning) is aimed to be built in an inclusive and iterating way in Japanese curricula. Notably, attending to the developmental nature of covariational reasoning (Carlson et al., 2002) mental actions involved in each covariational reasoning level has been revisited and deepened at successive grade level. Therefore, Japanese curricula illustrate a great model for spiral curriculum, especially for developing the proportional and functional relationships throughout the elementary and lower secondary level mathematics.

The development of the concepts of proportions and linear functions in the curricula are answering the calls of research. Particularly, in Japanese curricula, the concepts of functions are developed through covariational reasoning at first, then correspondence meaning is shared in the textbook after triggering the highest covariational reasoning level iteratively. Researchers argue that covariation meaning of functions are more relevant to students' daily engagement with the topic and their use of covariational perspective can lead to the development of correspondence view of functions (Thompson, 1994b; Carlson et al., 2002; Oehrtman et al., 2008; Smith, 2003; Lloyd et al., 2010). Moreover, as suggested in literature, functions are examined through different representations and dynamic situations (e.g., Thomson & Carlson, 2017); the task variables get more complex gradually (e.g., Heinz, 2000); the difference between functions and equations are presented (e.g., Chazan & Yerushalmy, 2003); the relationship between proportional relationships and linear functions are explained (e.g., Lloyd et al., 2011); the invariant relationship between variables are introduced as rate of change in dynamic situations (e.g., Carlson, 2002); real life situations are used to study proportional relationships and functions (e.g., Carlson et al., 2002; Oehrtman et al., 2008); quantities and quantitative operations are explicitly introduced (e.g., Thompson, 1994) in Japanese curricula. That is, Japanese curricula are designed in a way to bridge the mathematics education literature with teachers' teaching. Lastly, the findings of the study pointed out that Japanese curricula attentively focus on not only conceptual understanding but also mathematical thinking in a developmental way (e.g., successive development of mental actions). Hence, we suggest curriculum developers pay attention to the spiral nature of Japanese curricula. We argue that Japanese curricula can be used by teacher educators to study and improve teachers' covariational reasoning. Our analysis also supports Thompson and Carlson (2017)'s argument that Japanese

curricula exemplifies how quantitative and covariational reasoning can be integrated in standards and textbooks.

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THE TREATMENT OF LINEAR FUNCTIONS IN JAPANESE CURRICULA THROUGH QUANTITATIVE AND COVARIATIONAL REASONING

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This study investigated how Japanese curricula represent functional relationships through the lenses of quantitative and covariational reasoning. Utilizing both macro and micro textbook analyses, we examined the tasks, questions, and representations in the Japanese elementary and lower secondary course of study, teachers' guide, and textbooks. Findings showed that, starting from the 4th grade by the end of 8th grade, Japanese curricula focus on iteratively improving learners' quantitative and covariational reasoning gradually raising up to continuous covariation level. This pointed out that Japanese curricula have a spiral nature in the learning of functional relationships involved in proportional and linear situations. We discuss the implications of findings for teaching, learning, and teacher education.

Keywords: curriculum, quantitative reasoning, covariational reasoning, linear functions

Background

School curriculum has a role in improving students' learning and acvement (Cai, 2017). The term curriculum is a broad notion including intended, implemented, attained, and potentially implemented curriculum. *Intended curriculum* is a planned set of actions for students to achieve a specific goal. *Implemented curriculum* refers to how the goals of the intended curriculum are implemented within classrooms and the *attained curriculum* is considered as what is learned by students in classrooms. Within the span of curriculum research, textbooks are considered a potentially *implemented curriculum* as they build a pathway "… between the intentions of the designers of curriculum policy and the teachers that provide instruction in classrooms" (Valverde et al., 2002, p.2). Being mediators between intended and implemented curriculum, textbooks are considered as having a strong impact on what occurs in classrooms (Valverde et al., 2002). In addition, textbooks provide information about pedagogical predispositions, nature of subject matter, arrangements of topics, and level of complexity. Therefore, textbooks have been accepted as tools to be investigated (Cai, 2017) in order to gain some possible insights about opportunities for students to learn mathematics (e.g., Son & Senk, 2010), teacher's teaching and learning (e.g., Son & Kim, 2015), and educational preferences of a particular country (e.g., Usiskin, 2013). Son and Diletti (2017) further argued that teachers' guidebooks and other supplementary materials (e.g., assessments) might be classified as potentially implemented curriculum since they also have potential impact on teaching. It is in this regard that, in this study, we examined Japanese curricula including the course of study, teachers' guide and textbooks because of their potential impact on learning of mathematics topics and classroom teaching.

Textbook analysis is mainly done either on the overall structure of textbooks, which is named as macroanalysis (i.e., horizontal analysis) or on the treatment of specific mathematics topics, which is called as microanalysis (i.e., vertical analysis). Researchers recommended to use both the macro and micro analysis to investigate the treatment of a mathematics topic so that the topic could be examined in terms of its place in specific educational system, related topics to it, and learning opportunities provided in it (Charalambous et al., 2010; Watanabe et al., 2017; Li, 2000). Regarding microanalysis, the treatment of functions is one important notion to study

because functions are the backbone of the domain of algebra and lays the foundation for advanced mathematics (Kaput, 1994). Functional relationships between quantities can be modeled by linear, quadratic, polynomial, exponential functions etc. Although scarce in number (Son & Diletti, 2017), there are some studies focusing on secondary school mathematics such as quadratic functions (e.g., Sağlam & Alacacı, 2012) and linear functions (e.g., Wang et al., 2017). Particularly, Wang et al. (2017) examined mathematics textbooks in Shanghai and in England by looking at the hierarchical levels of students' understanding of linear functions through some aspects, such as dependent relationship and connecting representations. Functional relationships can mainly be expressed in two ways: (i) by a correspondence relationship between elements of two sets and (ii) by representing covariation in quantities (Blanton et al., 2011). Though, the topics of functions were suggested to be developed through quantitative and covariational reasoning and to do it starting with earlier grades rather than waiting until high school and college (e.g., Oehrtman et al., 2008). Thompson and Carlson (2017) defined:

A function, covariationally, is a conception of two quantities varying simultaneously such that there is an invariant relationship between their values that has the property that, in the person's conception, every value of one quantity determines exactly one value of the other. (p. 436)

Lloyd et al. (2010) pointed to the essentiality of understanding functions as means to outline how related quantities covary. In addition, studies showed that students who cannot reason variationally and covariationally seemed experiencing difficulties about functional and proportional relationships (e.g., Carlson et al., 2002). It is also found that undergraduate students start their collegiate study with weak understanding of functions, thus experiencing deficiency about making sense of dynamic events (e.g., Carlson et al., 2002). Therefore, research suggested designing instruction and curriculum for improving the learners' understanding of functional relationships (Thompson & Carlson, 2017; Carlson et al., 2002). Thus, in compliance with earlier research (e.g., Karagöz et al. 2022; Taşova et al., 2018), we acknowledge that quantitative and covariational reasoning might be a possible avenue for analyzing curricular materials. Similarly, Thompson and Carlson (2017) argue that Japanese curricula exemplifies how quantitative and covariational reasoning can be integrated in standards and textbooks.

It is in this regard that, in this study, we investigated the topic of functional relationships in Japanese curricula between the grades of 4 to 8 through the lenses of quantitative and covariational reasoning. We also particularly attended to the call for integrating macro and micro analyses to investigate the research questions of "How might Japanese curricula materials depicted in the topic of functional relationships potentially trigger quantitative reasoning and covariational reasoning? In what ways tasks, problem situations, questions, and the use of representations (including algebraic, graphical, tabular, and verbal) in functional relationships might potentially trigger quantitative and covariational reasoning in Grades 4 to 8?"

Theoretical Framework

Thompson (1990) defined quantitative reasoning as "the analysis of a situation into a quantitative structure—a network of quantities and quantitative relationships" (p. 12). A quantity is a measurable quality of an object coming into being with a person's conception of a situation by considering the measurable quality of an object (Thompson, 1994a). Both quantitative and covariational reasoning is about conceiving a situation where the former requires a person to conceive the situation in terms of a quantitative structure and the latter is necessary for a person to conceive the dynamic situation which involves quantities' varying values (Thompson &

Carlson, 2017). Carlson et al. (2002) defined covariational reasoning as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). They argued that these activities (i.e., images of covariation) are developmental, meaning that "the images of covariation can be defined by level and that the levels emerge in an ordered succession" (Carlson et al., 2002, p. 354). In their covariational reasoning framework, Carlson et al. presented five mental actions originating in five covariational reasoning levels (see Figure 1).

Covariational Reasoning Level	**Description of the Covariational Reasoning Level**	**Necessary Mental Actions for the reasoning level**	**Description of the highest mental action**
Smooth continuous covariation	"The person envisions increases and decreases (thereafter, changes) in one quantity's or variable's value, and the person envisions both variables varying smoothly and continuously." (Thompson and Carlson, 2017, p. 435)	MA5 MA4 MA3 MA2 MA1	**MA5:** "Coordinating the instantaneous rate of change of the functions with continuous changes in the independent variable for the entire domain of the function." (Carlson *et al.*, 2002, p. 357)
Chunky continuous covariation	"The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation." (Thompson and Carlson, 2017, p. 435)	MA4 MA3 MA2 MA1	**MA4:** "Coordinating the average rate-of-change of the function with uniform increments of change in the input variable." (Carlson *et al.*, 2002, p. 357)
Coordination of values	"The person coordinates the value of one variable (x) with values of another variable (y) with the anticipation of creating a discrete collection of pairs (x,y)." (Thompson and Carlson, 2017, p. 435)	MA3 MA2 MA1	**MA3:** "Coordinating the amount of change of one variable with changes in the other variable." (Carlson *et al.*, 2002, p. 357)
Gross coordination of values	"The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values." (Thompson and Carlson, 2017, p. 435)	MA2 MA1	**MA2:** "Coordinating the direction of change of one variable with changes in the other variable." (Carlson *et al.*, 2002, p. 357)
Precoordination of values	"The person envisions two variables' values varying, but asynchronously- one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects." (Thompson and Carlson, 2017, p. 435)	MA1	**MA1:** "Coordinating the value of one variable with changes in the other." (Carlson *et al.*, 2002, p. 357)
No coordination	"The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values." (Thompson and Carlson, 2017, p. 435)	-	-

Figure 1: Covariational reasoning levels and corresponding mental actions

Method

In this study we examined the Mathematics International (MI) textbook series published in 2011 globally by Tokyo Shoseki in collaboration with Global Educational Resources. Tokyo Shoseki is a leading textbook publisher in Japan and its textbook series are one of the six most widely used series in elementary mathematics (Watanabe et al., 2017). The MI series cover grades 1 to 9 which includes both elementary (grades 1 to 6) (Fujii & Iitaka, 2012) and lower secondary (grades 7 to 9) school mathematics (Fujii & Matano, 2012). In addition, we investigated the course of study (COS) published in 2008, which is the national curriculum standards in Japan, and teachers' guides, which is defined as "a guidebook on Japanese curriculum standards" (Isoda, 2010a, p. i) for teachers. Both were provided by the Ministry of National Education of Japan.

To analyze the curricula materials, we utilized a content analysis method (Weber, 1990). First, we determined the grade levels in the Japanese COS where the functional relationships were covered. We found out that under the topic of functional relationships, rate, ratio, proportion, and linear functions were covered between the grades 4 and 8. In this paper we specifically focus on linear functions and proportional relationships. We conducted our analysis in the order of the course of study (COS), teachers' guide, and the units of textbooks. Our unit(s) of analysis in all the curricular materials were a statement or a set of statements, graphs, diagrams, tables and symbols. Using both quantitative reasoning and covariational reasoning frameworks as our theoretical constructs, we examined how quantities were introduced and how the relationships between them were triggered in the statements, tasks, questions, problem

situations and representations throughout curricula. As covariational reasoning enables examining quantitative reasoning in dynamic situations, using the categories in Figure 1, we examined which mental actions were targeted on the part of students to engage in covariational reasoning. This way, we determined the tasks, questions, problem situations, and representations that seemed potentially to trigger the mental actions, and eventually covariational reasoning.

Findings

Findings has shown that the level of covariational reasoning and complexity of task variables increases with the increase in grade level (see Figure 2).

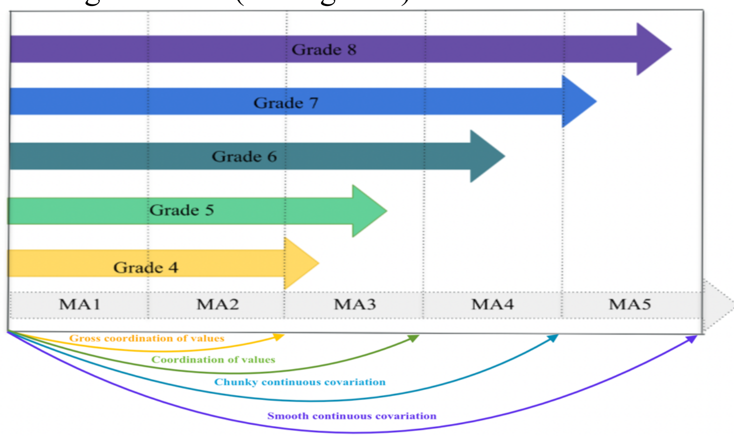

Figure 2: Spiral and iterative nature of Japanese curricula with respect to the metal actions in covariational reasoning

Particularly, in the context of functional relationships, the gross coordination level of covariational reasoning has been supported starting from the 4th grade to the 8th grade. Although the tasks and questions in each grade mostly focus on and delve into specific mental actions with the corresponding covariational reasoning level, there is a spiral nature such that the covariational reasoning level in the upcoming grade builds on and deepens on the previous ones. This suggests that mental actions in the previous covariational reasoning level are iteratively triggered to build the next level of reasoning. Particularly, *gross coordination of values* level of covariational reasoning seems to be aimed in the 4th grade by particularly triggering MA1 and MA2; *coordination of values* level seems to be targeted in the 5th grade by particularly triggering MA1, MA2, and MA3; and *continuous covariation* seems to be targeted in the 6th and 7th grades by particularly triggering MA1, MA2, MA3, and MA4; and (chunky and smooth) *continuous covariation* seems to be targeted in the 8th grade by particularly triggering MA1, MA2, MA3, MA4 and MA5. In lieu of space, we do not share detailed analysis, rather we aim to present different mental actions, reasoning levels, or representations that's not included in the previous grade.

In *4th grade*, in the COS, students are expected to "represent and investigate the relationship between two quantities as they vary simultaneously" (Takahashi et al., 2008, p. 11). Teachers are suggested to use "activities to find two quantities in everyday life that vary in proportion to each other, and to represent and investigate the relationships of numbers/ quantities in tables and graphs." (Isoda, 2010a, p. 116). Also, the Japanese curriculum explicitly differentiates quantities from numbers as stating "a quantity expresses size of an object…A number of objects can be expressed in integers, for example by counting them. On the other hand, in measuring length of

strings or weight of water, the quantity can be divided infinitely and cannot always be expressed integers" (Isoda, 2010, p.34). Particularly, teachers are suggested to present activities having quantities of actual objects compared to make it easier for students "to grasp what quantity is being compared, and the meaning of the quantity will become gradually clear to students …For length, for example, "long/short"; … for speed, "fast/slow" (Isoda, 2010, p. 35). All these suggested that there is emphasis on the fact that quantities are measurable qualities of objects such that speed measures whether an object is fast or slow while length measures whether an object is long or short. In the textbook, students are given a broken-line graph to represent monthly temperature changes of Tokyo to build the gross coordination of values level of covariational reasoning. Particularly, students are expected to verbally indicate the axes and coordinate the values, such as, the highest temperature and the corresponding specific month (MA1). They are also expected to focus on the direction of change (i.e., increase, decrease, constancy) of temperature in months (MA2). Thus, the goal in this grade seems to trigger *gross coordination of values* level of covariational reasoning.

In *5th grade*, the objectives and teachers' guide explanations aim for students to (i) deepen their understanding about covarying quantities, and (ii) consider and express quantitative relationships through using different representations. The main goal seems to deepen the *gross coordination of values* level; but also *coordination of amount of changes* of variables (MA3) has seemed to be introduced. In the textbook, through a task asking to use sticks of the same length side by side to make 30 squares, students are expected to recognize the pattern of an arithmetic sequence by examining the change in quantities and their relationships. As an example of two different students' ideas, the following figure is given in the textbook.

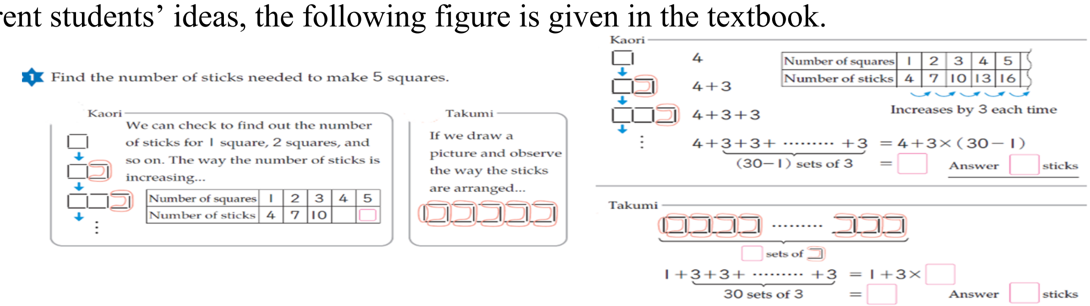

Figure 3: Students' ideas on stick and square task (Fujii and Iitaka, 2012, Grade 5, p. A103-104)

The variables are the number of squares and the number of sticks which are both discrete. A tabular representation is presented in Kaori' approach where she considers the changes between two consecutive values of the number of sticks for each square. However, Takumi examines the arrangement of sticks in the picture where his focus is on how many more sticks (i.e., 3 more sticks) is needed for each square given the first stick. Particularly, Kaori presented quantities, number of squares and number of sticks, in chunks at her table. Albeit the numerical values, her focus is on how the two quantities vary simultaneously such that the number of squares is increasing by one while the number of sticks is increasing by three each time. That is, students' attention is on the increments of 1 and 3 that quantify the increase in the number of squares and the number of sticks respectively. Also, students are expected to compare the differences (the increments of one and three) between the two quantities which might help determine the idea that for each difference (increment of one) in the number of squares, there is some other difference (increment of three) in the number of sticks. Moreover, students are expected to

verbalize what the values of 3, 4, 30, and (30-1) represent given the mathematical sentence $4 + 3 \times (30 - 1)$, an important connection between algebraic and verbal representations. In addition, Kaori's table seems to illustrate the correspondence of increase in number of squares with the increase in number of sticks, and her explanation supports that she considers "the way the number of sticks increasing…" (Fujii and Iitaka, 2012, Grade 5, p. A103). Therefore, Kaori's behavior seems to underscore MA3 and indicate her covariational reasoning level as *coordination of values*. On the contrary, Takumi does not seem to recognize that for each new square there are three more sticks needed. His way of thinking suggests that he seems to focus on the number of squares and the "increase" in the number of sticks rather than coordinating the amount of change in one quantity with the change in the other as Kaori. Takumi's behavior seems to underscore MA2 which corresponds to *gross coordination of values*.

In *6th grade*, students are expected to both (i) understand and investigate direct proportional relationships and (ii) gain an awareness of inversely proportional relationships by examining the situations that involve quantities varying simultaneously (Takahashi et al., 2008). The teachers' guide states that "the quotient of two corresponding quantities remains constant" (Isoda, 2010a, p. 155) in the direct proportional relationships which represent "a straight line passing through the origin." (Isoda, 2010a, p. 155). This characteristic is explicitly stated as important to distinguish proportional relationships. Teachers are suggested to use "activities such as representing on a graph two quantities that vary simultaneously so that they [students] understand that if two quantities are in a proportional relationship, the graph representing this relationship is a straight line" (Isoda, 2010a, p. 155)

The task below from 6th grade textbook shows that *coordination of values* level of covariation (up to MA3) aimed to be deepened and *continuous covariation* (MA4) seemed to be triggered. In lieu of space, we do not provide detailed analysis but our account of how students might possibly reason.

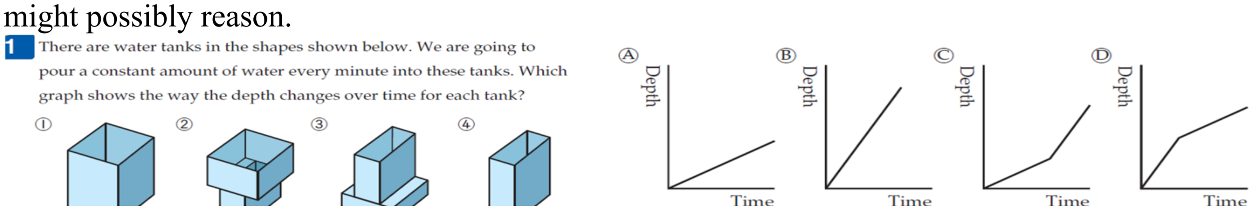

Figure 4: What Will the Graph Look Like (Fujii and Iitaka, 2012, Grade 6, p. B100)

In the task, two continuous variables, time and depth of water, are simultaneously compared through using graphical representations. Students might think that the narrower the base area of the tank the greater the amount of water poured into the tank in a minute. If a student thinks the coordination of depth with amount of time passing (M1), considers the increase in the depth of the water with regard to the time (MA2), compares the amounts of changes in depth of the water for some amount of time (MA3), and envisions the average rate of change of depth as increasing simultaneously with the time (MA4), the student might have *continuous covariation* level of covariational reasoning. However, the action of coordinating the rate of change with the uniform increments of inputs (MA4) are not examined through the numerical values of quantities. Thus, we claim that the task might be targeted to raise students' awareness about the rate of change as a scaffold for the upcoming grades rather than explicitly fostering MA4 in this task.

In *7th grade*, students are expected to further enhance their understanding of direct and inverse proportional relationships in real-life situations. Yet, differently from the 6th grade, the direct and inverse proportional relationships is planned to be re-examined through focusing on

the simple forms of linear functions $y = ax$ and reciprocal functions $y = \frac{a}{x}$ respectively (Isoda, 2010b). Differently from 6th grade, students are expected to examine similarities and differences between direct and inverse proportion graphs. Moreover, the rationale of the use of proportional relationships is explicitly stated as to trigger correspondence and covariation meaning of functions in many phenomena in daily life (Isoda, 2010b).

We share an example from the MI textbook showing direct and indirect proportional relationship between two variables and how the continuous nature of their covarying relationship is represented visually (see Figure 5).

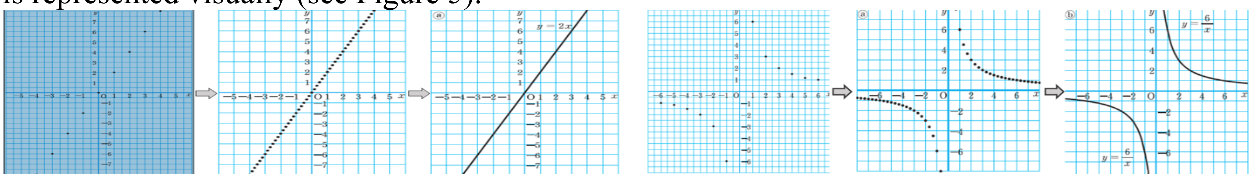

Figure 5: Graphing of y=2x (Fujii and Matano, 2012, Grade 7, p. 117-118) and graphing of $y = \frac{6}{x}$ (Fujii and Matano, 2012, Grade 7, pp. 127-129)

Different integer values of y and x are given for plotting both $y = 2x$ and $y = \frac{6}{x}$. Students are asked to explicitly think about the interval for x values and corresponding y values getting smaller and smaller producing a straight line or a curve. It seems that the continuous nature of variables is targeted through graphical representations. This way of representing might lay a foundation to smooth continuous covariation (corresponding up to MA5), although it is not explicitly introduced.

In *8th grade*, linear functions are introduced, and continuous covariation seems to be triggered and deepened. Students are expected to examine the changes and correspondences of two quantities again with a focus on graphical, algebraic, tabular, and verbal representations. In addition, the rate of change for the linear functions (i.e., in the form of $y = ax + b$) is expressed as $\frac{y_2 - y_1}{x_2 - x_1}$. Here, the rate of change which equals the constant a, is expressed as "how much y will increase when x increases by 1" (Isoda, 2010b, p. 92). All these focus on relationships between amounts of changes in variables suggested that at least MA3 is targeted. Moreover, there is an emphasis on the difference between the expressions of equations with two variables and functions. When the coordination of values of y and x is considered for an equation with two variables expressed as $ax + by + c = 0$, there is one and only one y value for every x value if $b \neq 0$. This relationship indicates that y is a function of x and the equation could be rewritten as $y = -\frac{a}{b}x - \frac{c}{b}$ to illustrate the functional relationship explicitly (Isoda, 2010b).

We present an example from the textbook (see Figure 6) in which the values of temperature are decimal numbers not increasing uniformly with uniform integer changes in time.

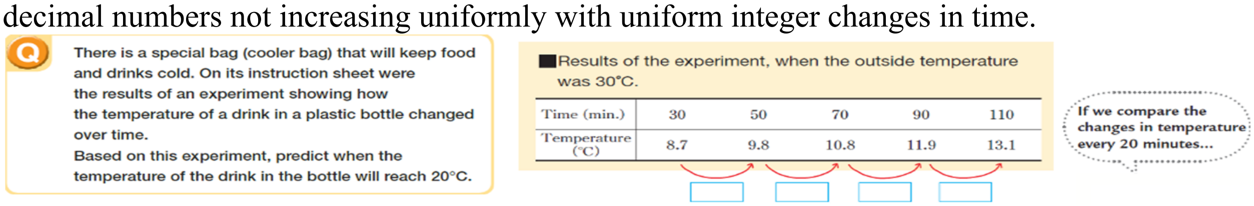

Figure 6: Temperature change of drinks task (Fujii and Matano, 2012, Grade 8, p. 70)

Students are asked to represent this situation with a graph on a coordinate system and think of the relationships between variables where temperature is a function of time. Students are

expected to use tables to express the result of values of variables and placing the values of temperature for corresponding units of time on a coordinate system (MA1), verbalize the direction of change as increase or decrease (MA2), coordinate the amount of change in y with the amount of increase in x (MA3), and coordinate the average rate of change of the temperature with uniform changes in time (MA4). Thus, students are supported to reason on the *chunky continuous covariation* level. Yet, in case students consider the rate of change as an instantaneous rate of change, MA5 might also be triggered. In 8th grade, students are also asked to compare linear functions (i.e., $y = ax + b$) with proportional equations (i.e., $y = ax$). It is explicitly explained that translation of the graph of $y = ax$ on the y axis by the value of b represents $y = ax + b$, where the value of b is expressed as y-intercept. Moreover, the use of different types of representations and the relationship between them are explicitly presented.

Discussion and Conclusion

The findings showed that proportional and functional relationships are presented as intertwined with the quantitative and covariational reasoning starting from 4th grade to 8th grade in Japanese curricula. Starting from gross coordination of values, the highest level of covariational reasoning (smooth covariational reasoning) is aimed to be built in an inclusive and iterating way in Japanese curricula. Notably, attending to the developmental nature of covariational reasoning (Carlson et al., 2002) mental actions involved in each covariational reasoning level has been revisited and deepened at successive grade level. Therefore, Japanese curricula illustrate a great model for spiral curriculum, especially for developing the proportional and functional relationships throughout the elementary and lower secondary level mathematics.

The development of the concepts of proportions and linear functions in the curricula are answering the calls of research. Particularly, in Japanese curricula, the concepts of functions are developed through covariational reasoning at first, then correspondence meaning is shared in the textbook after triggering the highest covariational reasoning level iteratively. Researchers argue that covariation meaning of functions are more relevant to students' daily engagement with the topic and their use of covariational perspective can lead to the development of correspondence view of functions (Thompson, 1994b; Carlson et al., 2002; Oehrtman et al., 2008; Smith, 2003; Lloyd et al., 2010). Moreover, as suggested in literature, functions are examined through different representations and dynamic situations (e.g., Thomson & Carlson, 2017); the task variables get more complex gradually (e.g., Heinz, 2000); the difference between functions and equations are presented (e.g., Chazan & Yerushalmy, 2003); the relationship between proportional relationships and linear functions are explained (e.g., Lloyd et al., 2011); the invariant relationship between variables are introduced as rate of change in dynamic situations (e.g., Carlson, 2002); real life situations are used to study proportional relationships and functions (e.g., Carlson et al., 2002; Oehrtman et al., 2008); quantities and quantitative operations are explicitly introduced (e.g., Thompson, 1994) in Japanese curricula. That is, Japanese curricula are designed in a way to bridge the mathematics education literature with teachers' teaching. Lastly, the findings of the study pointed out that Japanese curricula attentively focus on not only conceptual understanding but also mathematical thinking in a developmental way (e.g., successive development of mental actions). Hence, we suggest curriculum developers pay attention to the spiral nature of Japanese curricula. We argue that Japanese curricula can be used by teacher educators to study and improve teachers' covariational reasoning. Our analysis also supports Thompson and Carlson (2017)'s argument that Japanese

curricula exemplifies how quantitative and covariational reasoning can be integrated in standards and textbooks.

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